
A Formulation for the Boundary-Layer Equations in General Coordinates

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SUMMARY

This is a working paper in which a formulation is given for solving the boundary-layer equations in general body-fitted curvilinear coordinates while retaining the original Cartesian dependent variables. The solution procedure does not require that any of the coordinates be orthogonal, and much of the software developed for many Navier-Stokes schemes can be readily used. A limited number of calculations have been undertaken to validate the approach.

INTRODUCTION

The boundary-layer approximation is a useful engineering tool which has contributed significantly to the understanding of viscous flow at high Reynolds number. The boundary-layer equations require the use of a body conforming coordinate system and the flow Reynolds number must be high. In developing the usual boundary-layer equations, both the independent variables and the dependent velocity variables are transformed to the new body conforming coordinates. For body surfaces with little curvature, the boundary-layer equations cast in terms of the new dependent variables more or less simplify back to a flat plate or Cartesian-like-form of the equations along a developed surface. However, if the body has appreciable curvature, the equations become more complicated. They are particularly more complex if a nonorthogonal coordinate system is used, yet for many applications it is difficult to generate an orthogonal coordinate system along the body surface.

In this note, a formulation for the boundary-layer equations in terms of the original Cartesian velocity variables is described for body-fitted general curvilinear coordinates. Although the collaborating computational experiments that have been undertaken are limited, the proposed alternate form of the governing equations may offer several advantages in terms of numerical stability by avoiding coordinate source terms. Moreover, this alternate form of the boundary-layer equations does not require that any of the coordinates be orthogonal, and software (grids, boundary condition routines, etc.) developed for many Navier-Stokes schemes can be readily used.

This formulation was partially motivated by discussions with Dr. H. Yoshihara.

BACKGROUND

The compressible boundary-layer equations for the unsteady, three-dimensional flow of a perfect gas over a *flat plate* can be written

$$\rho_t + (\rho u)_x + (\rho v)_y + (\rho w)_z = 0 \quad (1a)$$

$$\rho u_t + \rho u u_x + \rho v u_y + \rho w u_z = -p_x + (\mu u_y)_y \quad (1b)$$

$$\rho w_t + \rho u w_x + \rho v w_y + \rho w w_z = -p_z + (\mu w_y)_y \quad (1c)$$

$$\begin{aligned} \rho H_t + \rho u H_x + \rho v H_y + \rho w H_z \\ = \left\{ \frac{\mu}{Pr} \left[H_y + \frac{Pr-1}{2} (u^2 + w^2)_y \right] \right\}_y + p_t \end{aligned} \quad (1d)$$

where ρ and p are the density and pressure, u, v, w are Cartesian velocity components, and H is the specific total enthalpy, $H = \frac{e+p}{\rho}$ with e the total energy per unit volume. Here Pr is the effective Prandtl number. The equations are nondimensionalized and y is a coordinate normal to the body surface. These equations can be used for bodies of slight surface curvature using x and z as distances along the surface with u, w , and v the corresponding velocities.

If surface curvature effects are taken into account, the boundary-layer equations take on a more complex form primarily because of the addition of coordinate source terms (refs. 1 and 2; Warsi, Z. U. A.; unpublished notes, 1984). For example, if ξ, η , and ζ are defined as shown in figure 1, with η normal to the surface, Panaras (ref. 2) derived the form

$$(J^{-1}\rho)_\tau + (J^{-1}\rho U)_\xi + (J^{-1}\rho V)_\eta + (J^{-1}\rho W)_\zeta = 0 \quad (2a)$$

$$\begin{aligned} \rho U_t + \rho U U_\xi + \rho V U_\eta + \rho W U_\zeta + K_{a1}\rho U^2 + K_{a2}\rho U W + K_{a3}\rho W^2 = \\ K_{a4}p_\zeta - K_{a5}p_\xi + \mu K_{a8}U_\eta + \mu K_{a9}W_\eta \\ + J \partial_\eta \left[\frac{\mu}{g_{22}J} (U_\eta + K_{a6}U + K_{a7}W) \right] \end{aligned} \quad (2b)$$

$$\begin{aligned} \rho W_t + \rho U W_\xi + \rho V W_\eta + \rho W W_\zeta + K_{b1}\rho U^2 + K_{b2}\rho U W + K_{b3}\rho W^2 = \\ K_{b4}p_\xi - K_{b5}p_\zeta + \mu K_{b8}U_\eta + \mu K_{b9}W_\eta \\ + J \partial_\eta \left[\frac{\mu}{g_{22}J} (W_\eta + K_{b6}U + K_{b7}W) \right] \end{aligned} \quad (2c)$$

$$\begin{aligned} (J^{-1}e)_\tau + [J^{-1}(e+p)U]_\xi + [J^{-1}(e+p)V]_\eta + [J^{-1}(e+p)W]_\zeta = \\ \partial_\eta [\mu K_{e1} \partial_\eta U^2 + \mu K_{e2} \partial_\eta (UW) + \mu K_{e3} \partial_\eta W^2 + \frac{\mu}{Pr(\gamma-1)g_{22}J} (a^2)_\eta] \end{aligned} \quad (2d)$$

where the metrics are given in appendix 1 and U and W are contravariant velocities as defined later. Here the η -coordinate must be normal to the body surface.

Although there are more terms than previously, this is a fairly clean form of the boundary-layer equations. From a numerical point of view, however, equation (2) are more difficult to solve than equation (1) because derivatives of metrics must be formed which are not always smooth and because the extra source terms can adversely effect numerical stability. For example, any coordinate source term such as $K_{a1}\rho U^2$ in equation (2b) can degrade stability, in this case if $K_{a1}U$ has a negative value.

In the following section the steps to bring the Navier-Stokes equations in Cartesian coordinates into general curvilinear coordinates and velocity components are begun as if to derive equations like equation (2). Here these steps are only taken to determine how to group the equations. Once the pressure in the viscous layer is determined from the uncoupled normal-like momentum equation, primitive forms of the momentum equations are used in place of equations 2b and 2c. The primitive forms with given pressure and use of a thin-layer approximation are easier to treat numerically than equation (2).

DEVELOPMENT OF CURVILINEAR BOUNDARY-LAYER EQUATIONS

The three-dimensional Navier-Stokes equations in Cartesian coordinates can be written as

$$(\rho)_t + (\rho u)_x + (\rho v)_y + (\rho w)_z = 0 \quad (3a)$$

$$\rho u_t + \rho u u_x + \rho v u_y + \rho w u_z + p_x = R_{xmom} \quad (3b)$$

$$\rho v_t + \rho u v_x + \rho v v_y + \rho w v_z + p_y = R_{ymom} \quad (3c)$$

$$\rho w_t + \rho u w_x + \rho v w_y + \rho w w_z + p_z = R_{zmom} \quad (3d)$$

$$\rho H_t + \rho u H_x + \rho v H_y + \rho w H_z - p_t = R_{ener} \quad (3e)$$

where $(R_{xmom}, R_{ymom}, R_{zmom})$ and R_{ener} will represent the viscous terms.

Transforming the independent variables $x, y,$ and z to body conforming coordinates $\xi, \eta,$ and ζ gives

$$(J^{-1}\rho)_\tau + (J^{-1}\rho U)_\xi + (J^{-1}\rho V)_\eta + (J^{-1}\rho W)_\zeta = 0 \quad (4a)$$

$$\rho u_t + \rho U u_\xi + \rho V u_\eta + \rho W u_\zeta + (\xi_x p_\xi + \eta_x p_\eta + \zeta_x p_\zeta) = R_{xmom} \quad (4b)$$

$$\rho v_t + \rho U v_\xi + \rho V v_\eta + \rho W v_\zeta + (\xi_y p_\xi + \eta_y p_\eta + \zeta_y p_\zeta) = R_{ymom} \quad (4c)$$

$$\rho w_t + \rho U w_\xi + \rho V w_\eta + \rho W w_\zeta + (\xi_z p_\xi + \eta_z p_\eta + \zeta_z p_\zeta) = R_{zmom} \quad (4d)$$

$$\rho H_t + \rho U H_\xi + \rho V H_\eta + \rho W H_\zeta - p_t = R_{ener} \quad (4e)$$

where the $U, V,$ and W represent unscaled contravariant velocities; e.g.,

$$U = \xi_x u + \xi_y v + \xi_z w \quad (5a)$$

$$V = \eta_x u + \eta_y v + \eta_z w \quad (5b)$$

$$W = \zeta_x u + \zeta_y v + \zeta_z w \quad (5c)$$

and J is the transform Jacobian

$$J = (x_\xi y_\eta z_\zeta + x_\zeta y_\xi z_\eta + x_\eta y_\zeta z_\xi - x_\xi y_\zeta z_\eta - x_\eta y_\xi z_\zeta - x_\zeta y_\eta z_\xi)^{-1}$$

and other metrics are defined at the end of appendix 1. Note that the Cartesian velocity variables $u, v,$ and w are still retained as dependent variables and that the momentum equations are still the Cartesian momentum equations.

To obtain equations such as equation 2 using new dependent velocity variables U, V , and W , the dependent velocity variables can be transformed by taking linear combinations of the Cartesian momentum equations. Specifically these equations can be multiplied by the matrix \bar{C} defined as

$$\bar{C} = \begin{bmatrix} \bar{\xi}_x & \bar{\xi}_y & \bar{\xi}_z \\ \bar{\eta}_x & \bar{\eta}_y & \bar{\eta}_z \\ \bar{\zeta}_x & \bar{\zeta}_y & \bar{\zeta}_z \end{bmatrix} \quad (6)$$

where the overbar implies scaling of the metrics such that $\bar{\xi}_x = \xi_x / \sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}$, $\bar{\eta}_x = \eta_x / \sqrt{\eta_x^2 + \eta_y^2 + \eta_z^2}$ etc. Multiplying the three momentum equations by \bar{C} gives

$$\rho \bar{U}_t + \rho U \bar{U}_\xi + \rho V \bar{U}_\eta + \rho W \bar{U}_\zeta + s_\xi + \bar{\nabla} \bar{\xi} \cdot (\nabla \xi p_\xi + \nabla \eta p_\eta + \nabla \zeta p_\zeta) = \bar{\nabla} \bar{\xi} \cdot \vec{R} \quad (7a)$$

$$\rho \bar{V}_t + \rho U \bar{V}_\xi + \rho V \bar{V}_\eta + \rho W \bar{V}_\zeta + s_\eta + \bar{\nabla} \bar{\eta} \cdot (\nabla \xi p_\xi + \nabla \eta p_\eta + \nabla \zeta p_\zeta) = \bar{\nabla} \bar{\eta} \cdot \vec{R} \quad (7b)$$

$$\rho \bar{W}_t + \rho U \bar{W}_\xi + \rho V \bar{W}_\eta + \rho W \bar{W}_\zeta + s_\zeta + \bar{\nabla} \bar{\zeta} \cdot (\nabla \xi p_\xi + \nabla \eta p_\eta + \nabla \zeta p_\zeta) = \bar{\nabla} \bar{\zeta} \cdot \vec{R} \quad (7c)$$

where $\vec{R} = (R_{xmom}, R_{ymom}, R_{zmom})^t$ and where ∇ is the gradient operator. Terms like $\bar{\nabla} \bar{\xi} \cdot \nabla \zeta p_\zeta$ should be interpreted as $(\bar{\xi}_x \zeta_x + \bar{\xi}_y \zeta_y + \bar{\xi}_z \zeta_z) p_\zeta$.

The terms s_ξ , s_η , and s_ζ are coordinate source terms given as

$$\begin{aligned} s_\xi = & -\rho U (u \partial_\xi \bar{\xi}_x + v \partial_\xi \bar{\xi}_y + w \partial_\xi \bar{\xi}_z) \\ & -\rho V (u \partial_\eta \bar{\xi}_x + v \partial_\eta \bar{\xi}_y + w \partial_\eta \bar{\xi}_z) \\ & -\rho W (u \partial_\zeta \bar{\xi}_x + v \partial_\zeta \bar{\xi}_y + w \partial_\zeta \bar{\xi}_z) \end{aligned} \quad (8a)$$

$$\begin{aligned} s_\eta = & -\rho U (u \partial_\xi \bar{\eta}_x + v \partial_\xi \bar{\eta}_y + w \partial_\xi \bar{\eta}_z) \\ & -\rho V (u \partial_\eta \bar{\eta}_x + v \partial_\eta \bar{\eta}_y + w \partial_\eta \bar{\eta}_z) \\ & -\rho W (u \partial_\zeta \bar{\eta}_x + v \partial_\zeta \bar{\eta}_y + w \partial_\zeta \bar{\eta}_z) \end{aligned} \quad (8b)$$

$$\begin{aligned} s_\zeta = & -\rho U (u \partial_\xi \bar{\zeta}_x + v \partial_\xi \bar{\zeta}_y + w \partial_\xi \bar{\zeta}_z) \\ & -\rho V (u \partial_\eta \bar{\zeta}_x + v \partial_\eta \bar{\zeta}_y + w \partial_\eta \bar{\zeta}_z) \\ & -\rho W (u \partial_\zeta \bar{\zeta}_x + v \partial_\zeta \bar{\zeta}_y + w \partial_\zeta \bar{\zeta}_z) \end{aligned} \quad (8c)$$

The Cartesian velocities in these source terms can be replaced with contravariant velocities using

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix} \quad (9)$$

where the transformation matrix is simply C^{-1} for C unscaled. Often the source terms are rearranged and written with Christoffel symbols, but such notation is not needed here.

In the body conforming coordinates, the wall corresponds to $\eta = \text{const}$, so $\nabla\eta$ is normal to the wall and the flow tangency condition requires that $\bar{V} = \overline{\nabla\eta} \cdot \vec{q} = 0$ where $\vec{q} = (u, v, w)^t$. For a thin viscous layer near the wall, \bar{V} will be $O(\delta)$ (as will the variation of ξ and ζ metrics with η , i.e. $\partial_\eta \xi_x = O(\delta)$ etc.). For small \bar{V} the η -momentum, equation 7b, simplifies to

$$s_\eta + \overline{\nabla\eta} \cdot (\nabla\xi p_\xi + \nabla\eta p_\eta + \nabla\zeta p_\zeta) = \overline{\nabla\eta} \cdot \vec{R}$$

Since \vec{R} represents viscous stress terms along the wall, the product $\overline{\nabla\eta} \cdot \vec{R}$ should be zero at the wall and small away from the wall. Consequently the η -momentum equation further simplifies to

$$s_\eta + \overline{\nabla\eta} \cdot (\nabla\xi p_\xi + \nabla\eta p_\eta + \nabla\zeta p_\zeta) = 0 \quad (10)$$

Given outer-edge conditions for p , this represents a simple wave equation with a source that can be used to evaluate the pressure through the viscous layer. This equation further simplifies if the η coordinate is orthogonal to the surface as $\overline{\nabla\eta} \cdot \nabla\xi = 0$ and $\overline{\nabla\eta} \cdot \nabla\zeta = 0$. Even for nonorthogonal coordinates, the coefficient $\overline{\nabla\eta} \cdot \nabla\eta$ is generally so much larger than the other coefficients that the assumption

$$p_\eta = 0 \quad (11)$$

is often valid over a thin layer near the wall. Thus from equation (11) p is prescribed throughout the boundary layer to its specified edge value. Therefore, pressure does not depend on the other dependent variables within the boundary layer if equation (11) is used, while the dependency is weak if equation (10) is used.

Once the pressure is determined in the boundary layer from the η momentum equation, the pressure derivatives in the Cartesian momentum equations, equations 4b-d can be evaluated. With pressure specified, the inviscid portion of the Cartesian momentum equations using only transformed independent variables are very easy to solve for u, v , and w . The inviscid equations with specified pressure are just simple convection equations. Consequently, assuming the thin-layer viscous terms cause no difficulties, equations 4b-d can be readily solved for u, v , and w in place of the more complex equations 7a and 7c for U and W . Taking a linear combination of $(u, v, w)^t$ with $\nabla\xi$ and $\nabla\zeta$ (now ignoring metric scaling), the transformed velocities U and W are then readily formed as

$$U = \nabla\xi \cdot \vec{q}$$

$$W = \nabla\zeta \cdot \vec{q}$$

At this point the same linear combination of Cartesian momentum equations is used to predict U and W , but the Cartesian momentum equations rather than the ξ and ζ momentum equations are used directly. As a consequence the complex source terms are avoided.

Up to this point the viscous terms have been mostly ignored in the development. Their complexity remains to be checked, as well as whether they tend to couple the equations together.

To examine the viscous terms, it is illustrative to first consider incompressible flow. For incompressible flow in which the coefficient of viscosity is constant the viscous terms are given as

$$R_{xmom} = \mu \nabla^2 u \quad (12a)$$

$$R_{ymom} = \mu \nabla^2 v \quad (12b)$$

$$R_{zmom} = \mu \nabla^2 w \quad (12c)$$

Clearly these terms do not couple the momentum equations together. Now the Laplacian operator transforms (in divergence form) as

$$\begin{aligned}\nabla^2 = J[& \partial_\xi J^{-1}(\nabla\xi \cdot \nabla\xi) \partial_\xi + \partial_\xi J^{-1}(\nabla\xi \cdot \nabla\eta) \partial_\eta + \partial_\xi J^{-1}(\nabla\xi \cdot \nabla\zeta) \partial_\zeta \\ & + \partial_\eta J^{-1}(\nabla\eta \cdot \nabla\xi) \partial_\xi + \partial_\eta J^{-1}(\nabla\eta \cdot \nabla\eta) \partial_\eta + \partial_\eta J^{-1}(\nabla\eta \cdot \nabla\zeta) \partial_\zeta \\ & + \partial_\zeta J^{-1}(\nabla\zeta \cdot \nabla\xi) \partial_\xi + \partial_\zeta J^{-1}(\nabla\zeta \cdot \nabla\eta) \partial_\eta + \partial_\zeta J^{-1}(\nabla\zeta \cdot \nabla\zeta) \partial_\zeta]\end{aligned}\quad (13)$$

The extra cross derivative terms caused by coordinate transformation have been encountered in potential and Navier-Stokes codes and can be differenced so as not to cause numerical instabilities. However, equation (13) can be further simplified by making a thin-layer approximation in which case these terms can be eliminated altogether. Dropping all derivatives with respect to ξ and ζ , the Laplacian is reduced to

$$\nabla^2 = J[\partial_\eta J^{-1}(\nabla\eta \cdot \nabla\eta) \partial_\eta] \quad (14)$$

which is particularly simple. Thus for an incompressible viscous term and specified pressure, the momentum equations (4b-d) are uncoupled and are readily solved for u , v , and w .

If the Cartesian form of the compressible viscous terms undergoes independent variable transformation, (x, y, z) to (ξ, η, ζ) , and if they are subject to a thin-layer approximation, they simplify to

$$R_{xmom} = J\partial_\eta[J^{-1}(\mu m_1 u_\eta + (\mu/3)m_2 \eta_x)] \quad (15a)$$

$$R_{ymom} = J\partial_\eta[J^{-1}(\mu m_1 v_\eta + (\mu/3)m_2 \eta_y)] \quad (15b)$$

$$R_{zmom} = J\partial_\eta[J^{-1}(\mu m_1 w_\eta + (\mu/3)m_2 \eta_z)] \quad (15c)$$

$$R_{ener} = J\partial_\eta\{J^{-1}[\mu m_1 m_3 + (\mu/3)m_2(\eta_x u + \eta_y v + \eta_z w)]\} \quad (15d)$$

where $m_1 = \eta_x^2 + \eta_y^2 + \eta_z^2$, $m_2 = \eta_x u_\eta + \eta_y v_\eta + \eta_z w_\eta$, and $m_3 = [(u^2 + v^2 + w^2)_\eta]/2 + Pr^{-1}(\gamma - 1)^{-1}(a^2)_\eta$. and J is again the transformation Jacobian. Here the viscous terms were first put into divergence form, and then simplified. As a result they are identical to the viscous terms used in many thin-layer Navier-Stokes codes. This can cause some error for bodies with high curvature as metric terms are also being discarded. Making the thin-layer approximation on the nonconservative form of the viscous terms eliminates this problem. This form is given in appendix 2.

The compressible viscous terms are coupled through the m_2 term, but the coupling appears to be weak and has been treated explicitly in many numerical schemes without degradation of the time step or iterative convergence. Moreover, because the coupling occurs in only one direction, η , it would not be too costly to account for it in a fully implicit manner.

CURVILINEAR BOUNDARY-LAYER EQUATIONS

In summary, a form of the boundary-layer equations for general curvilinear coordinates is given by (with η chosen away from the surface):

Normal Momentum

$$p_\eta = 0 \quad (11)$$

ξ and ζ Momentum

$$\rho u_t + \rho U u_\xi + \rho V u_\eta + \rho W u_\zeta + (\xi_x p_\xi + \eta_x p_\eta + \zeta_x p_\zeta) = J \partial_\eta \{ J^{-1} [\mu m_1 u_\eta + (\mu/3) m_2 \eta_x] \} \quad (4b)$$

$$\rho v_t + \rho U v_\xi + \rho V v_\eta + \rho W v_\zeta + (\xi_y p_\xi + \eta_y p_\eta + \zeta_y p_\zeta) = J \partial_\eta \{ J^{-1} [\mu m_1 v_\eta + (\mu/3) m_2 \eta_y] \} \quad (4c)$$

$$\rho w_t + \rho U w_\xi + \rho V w_\eta + \rho W w_\zeta + (\xi_z p_\xi + \eta_z p_\eta + \zeta_z p_\zeta) = J \partial_\eta \{ J^{-1} [\mu m_1 w_\eta + (\mu/3) m_2 \eta_z] \} \quad (4d)$$

where two linear combinations of u , v , and w are used

$$U = \nabla \xi \cdot \vec{q}$$

$$W = \nabla \zeta \cdot \vec{q}$$

Energy

$$\rho H_t + \rho u H_\xi + \rho v H_\eta + \rho w H_\zeta - p_t = J \partial_\eta \{ J^{-1} [\mu m_1 m_3 + (\mu/3) m_2 (\eta_x u + \eta_y v + \eta_z w)] \} \quad (4e)$$

where

$$m_1 = \eta_x^2 + \eta_y^2 + \eta_z^2$$

$$m_2 = \eta_x u_\eta + \eta_y v_\eta + \eta_z w_\eta$$

$$m_3 = [(u^2 + v^2 + w^2)_\eta] / 2 + Pr^{-1} (\gamma - 1)^{-1} (a^2)_\eta$$

Constitutive

$$\frac{\rho}{\rho_\infty} = \frac{p T_\infty}{T p_\infty} \quad (16a)$$

where

$$\frac{T}{T_\infty} = \frac{(\gamma - 1)}{a_\infty^2} \left[H - \frac{(u^2 + v^2 + w^2)}{2} \right] \quad (16b)$$

Continuity

$$(J^{-1} \rho)_\tau + (J^{-1} \rho U)_\xi + (J^{-1} \rho V)_\eta + (J^{-1} \rho W)_\zeta = 0 \quad (4a)$$

The six equations, normal momentum (eq. 11), two linear combinations of (eq. 4b), (4c), and (4d) to form U and W , energy (eq. 4e), state (eq. 16a), and continuity (4a), can be used to determine the six variables p, U, W, H, ρ , and V . The Cartesian velocity components are then obtained from

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix} \quad (9)$$

Generally p is determined from

$$p_\eta = 0 \quad (11)$$

and an outer-edge boundary condition so that $p = p_{edge}$ along η coordinates. In more general cases pressure can be determined from

$$s_\eta + \nabla \eta \cdot (\nabla \xi p_\xi + \nabla \eta p_\eta + \nabla \zeta p_\zeta) = 0 \quad (10)$$

using the same outer-edge condition with the source term defined as

$$\begin{aligned} s_\eta = & -\rho U (u \partial_\xi \bar{\eta}_x + v \partial_\xi \bar{\eta}_y + w \partial_\xi \bar{\eta}_z) \\ & -\rho V (u \partial_\eta \bar{\eta}_x + v \partial_\eta \bar{\eta}_y + w \partial_\eta \bar{\eta}_z) \\ & -\rho W (u \partial_\zeta \bar{\eta}_x + v \partial_\zeta \bar{\eta}_y + w \partial_\zeta \bar{\eta}_z) \end{aligned} \quad (8b)$$

In this case pressure will vary throughout the boundary layer.

For ζ (instead of η) away from the surface, linear combinations of U and V are used, $p_\zeta = 0$, $m_1 = \zeta_x^2 + \zeta_y^2 + \zeta_z^2$, $m_2 = \zeta_x u_\zeta + \zeta_y v_\zeta + \zeta_z w_\zeta$, $m_3 = [(u^2 + v^2 + w^2)_\zeta]/2 + Pr^{-1}(\gamma - 1)^{-1}(a^2)_\zeta$, and viscous derivatives are taken with respect to ζ .

NUMERICAL TESTING

This formulation has been tested in steady state applications using the time-like boundary-layer scheme reported by Van Dalsem and Steger in (ref. 3). For a prescribed edge pressure, the equations are solved in the following way with $p_\eta = 0$. Using central differencing in η and upwind differencing in ξ and ζ , equations (4b) to (4e) are used to update u, v , and w , and H . As pressure was already obtained using one linear combination of the momentum equations, only two linear combinations of the momentum equations can be used to determine the velocities. Thus, U and W are formed from the momentum updated u, v , and w . The equation of state, (16a), is used to update ρ , with T defined from equation (16b). The third linear combination of u, v , and w is obtained by solving continuity for V using already updated U, W , and ρ . The final updated form of the Cartesian velocity components are then obtained from updated U, V , and W using equation (9).

Two test cases were used to verify the algorithm. The first case, flow over a flat plate, was simply used to verify that the algorithm was coded correctly. To bring in three-dimensional effects and to verify some of the metric terms, the computational grid was rotated on the flat plate as sketched in figure 2. In this simple test, a compressible Blasius solution was specified at all boundaries. Figure 3 shows the Blasius result and a typical computed profile (with $Pr = 1$) taken from the center of the $20 \times 10 \times 30$ grid for a grid that was not rotated. Significant relative error is detected for the vertical velocity, but the overall error is small. Figure 4 shows Cartesian velocity profiles for a similar calculation but with the grid rotated by 30° .

A more interesting test case was provided by using the boundary-layer algorithm to verify a computed thin-layer Navier-Stokes result on a prolate spheroid (ref. 4). In this case the Navier-Stokes result was first obtained for the body at 10° angle of attack, $M_\infty = 0.17$, and $Re = 7.7 \times 10^6$ based on the body length. In the experiment (ref. 5) the boundary layer was tripped at the $x/L = 20\%$ station. In the calculation the trip was somewhat taken into account by using laminar values of the viscosity coefficient over the first 20% portion of the body and using an eddy viscosity turbulence model from that point on. Having obtained the Navier-Stokes result, the new boundary-layer code was also run on a portion of the Navier-Stokes grid from $x/L = 5\%$ to $x/L = 80\%$. Starting boundary-layer profiles and edge conditions were taken from the Navier-Stokes calculation, the "edge" corresponding to the 25th grid point up from the wall of the Navier-Stokes grid (about 0.003 body lengths). Various profiles from this calculation are shown in figures 5 and 6 at $x/L = 0.48$ and $x/L = 0.64$. Also shown in figure 7 are boundary-layer-computed wall-turning angles, Navier-Stokes-computed wall-turning angles, and experimental data (ref. 5). Although discrepancies are evident, the overall agreement is very good on the windward portion of the body and is more than adequate to verify the boundary-layer equations and algorithm.

CONCLUDING REMARKS

A formulation has been given for the boundary-layer equations using general body-fitted curvilinear coordinates and retaining the original Cartesian dependent variables so that coordinate source terms are avoided. The formulation does not require that any of the coordinates be orthogonal and gridding and software developed for many Navier-Stokes schemes can be readily used.

The curvilinear boundary-layer equations given previously have obvious similarity to the thin layer Navier-Stokes equations. However, in this boundary layer formulation, while three momentum equations are solved for three Cartesian velocity components, only two linear combinations of velocity variables are actually taken from momentum. A third linear combination of the momentum equations is used to provide the simplified η -momentum equation with a prescribed edge pressure. Pressure throughout the boundary layer thus is specified and uncoupled (or weakly coupled if eq. (10) is used over eq. (11)) from the other dependent variables within the boundary layer. This then allows the use of boundary-layer solution algorithms.

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APPENDIX 1

METRICS

The metric relations needed for the Panaras curvilinear boundary layer equations are given as:

$$K_{a1} = 0.5 g^{11} \frac{\partial g_{11}}{\partial \xi} + g^{13} \frac{\partial g_{13}}{\partial \xi} - 0.5 g^{13} \frac{\partial g_{11}}{\partial \zeta}$$

$$K_{a2} = g^{11} \frac{\partial g_{11}}{\partial \zeta} + g^{13} \frac{\partial g_{33}}{\partial \xi}$$

$$K_{a3} = 0.5 g^{13} \frac{\partial g_{33}}{\partial \zeta} + g^{11} \frac{\partial g_{13}}{\partial \zeta} - 0.5 g^{11} \frac{\partial g_{33}}{\partial \xi}$$

$$K_{a4} = \frac{g_{22} g_{13}}{g}$$

$$K_{a5} = \frac{g_{22} g_{33}}{g}$$

$$K_{a6} = 0.5 g^{11} \frac{\partial g_{11}}{\partial \eta} + 0.5 g^{13} \frac{\partial g_{13}}{\partial \eta}$$

$$K_{a7} = 0.5 g^{11} \frac{\partial g_{11}}{\partial \eta} + 0.5 g^{13} \frac{\partial g_{33}}{\partial \eta}$$

$$K_{a8} = g^{22} K_{a6}$$

$$K_{a9} = g^{22} K_{a7}$$

$$K_{b1} = 0.5 g^{13} \frac{\partial g_{11}}{\partial \xi} + g^{33} \frac{\partial g_{13}}{\partial \xi} - 0.5 g^{33} \frac{\partial g_{11}}{\partial \zeta}$$

$$K_{b2} = g^{13} \frac{\partial g_{11}}{\partial \zeta} + g^{33} \frac{\partial g_{33}}{\partial \xi}$$

$$K_{b3} = 0.5 g^{33} \frac{\partial g_{33}}{\partial \zeta} + g^{13} \frac{\partial g_{13}}{\partial \zeta} - 0.5 g^{13} \frac{\partial g_{33}}{\partial \xi}$$

$$K_{b4} = \frac{g_{11} g_{13}}{g}$$

$$K_{b5} = \frac{g_{11} g_{22}}{g}$$

$$K_{b6} = 0.5 g^{13} \frac{\partial g_{11}}{\partial \eta} + 0.5 g^{33} \frac{\partial g_{13}}{\partial \eta}$$

$$K_{b7} = 0.5 g^{13} \frac{\partial g_{13}}{\partial \eta} + 0.5 g^{33} \frac{\partial g_{33}}{\partial \eta}$$

$$K_{b8} = g^{22} K_{b6}$$

$$K_{b9} = g^{22} K_{b7}$$

$$K_{e1} = \frac{g_{11}}{2 g_{22}}$$

$$K_{e2} = \frac{g_{33}}{g_{22}}$$

$$K_{e3} = \frac{g_{33}}{2 g_{22}}$$

$$g_{11} = x_{\xi} x_{\xi} + y_{\xi} y_{\xi} + z_{\xi} z_{\xi}$$

$$g_{22} = x_{\eta} x_{\eta} + y_{\eta} y_{\eta} + z_{\eta} z_{\eta}$$

$$g_{33} = x_{\zeta} x_{\zeta} + y_{\zeta} y_{\zeta} + z_{\zeta} z_{\zeta}$$

$$g_{12} = x_{\xi} x_{\eta} + y_{\xi} y_{\eta} + z_{\xi} z_{\eta}$$

$$g_{13} = x_{\xi} x_{\zeta} + y_{\xi} y_{\zeta} + z_{\xi} z_{\zeta}$$

$$g_{23} = x_{\eta} x_{\zeta} + y_{\eta} y_{\zeta} + z_{\eta} z_{\zeta}$$

$$g^{11} = (g_{22} g_{33} - g_{23} g_{23}) / g$$

$$g^{22} = (g_{11} g_{33} - g_{13} g_{13}) / g$$

$$g^{33} = (g_{22} g_{11} - g_{12} g_{12}) / g$$

$$g^{12} = (g_{23} g_{31} - g_{21} g_{33}) / g$$

$$g^{13} = (g_{21} g_{32} - g_{22} g_{13}) / g$$

$$g^{23} = (g_{21} g_{13} - g_{11} g_{23}) / g$$

and

$$g = (\sqrt{J})^{-1}$$

The metrics used throughout the paper are defined as

$$\xi_x = J(y_{\eta} z_{\zeta} - y_{\zeta} z_{\eta})$$

$$\xi_y = J(x_\zeta z_\eta - x_\eta z_\zeta)$$

$$\xi_z = J(x_\eta y_\zeta - x_\zeta y_\eta)$$

$$\eta_x = J(y_\zeta z_\xi - y_\xi z_\zeta)$$

$$\eta_y = J(x_\xi z_\zeta - x_\zeta z_\xi)$$

$$\eta_z = J(x_\zeta y_\xi - x_\xi y_\zeta)$$

$$\zeta_x = J(y_\xi z_\eta - y_\eta z_\xi)$$

$$\zeta_y = J(x_\eta z_\xi - x_\xi z_\eta)$$

$$\zeta_z = J(x_\xi y_\eta - x_\eta y_\xi)$$

with

$$J = (x_\xi y_\eta z_\zeta + x_\zeta y_\xi z_\eta + x_\eta y_\zeta z_\xi - x_\xi y_\zeta z_\eta - x_\eta y_\xi z_\zeta - x_\zeta y_\eta z_\xi)^{-1}$$

APPENDIX 2

THIN-LAYER VISCOUS TERMS

Let the Cartesian form of the compressible viscous terms undergo independent variable transformation, (x, y, z) to (ξ, η, ζ) in chain rule conservative form. If they are subject to a thin layer approximation with ζ taken as the direction away from the body, they simplify to

$$\begin{aligned} R_{xmom} = & \zeta_x \partial_\zeta [\mu(\zeta_x u_\zeta + \zeta_x u_\zeta) + \lambda(\zeta_x u_\zeta + \zeta_y v_\zeta + \zeta_z w_\zeta)] \\ & + \zeta_y \partial_\zeta [\mu(\zeta_y u_\zeta + \zeta_x v_\zeta)] \\ & + \zeta_z \partial_\zeta [\mu(\zeta_z u_\zeta + \zeta_x w_\zeta)] \end{aligned}$$

$$\begin{aligned} R_{ymom} = & \zeta_x \partial_\zeta [\mu(\zeta_y u_\zeta + \zeta_x v_\zeta)] \\ & + \zeta_y \partial_\zeta [\mu(\zeta_y v_\zeta + \zeta_y v_\zeta) + (\zeta_x u_\zeta + \zeta_y v_\zeta + \zeta_z w_\zeta)] \\ & + \zeta_z \partial_\zeta [\mu(\zeta_z v_\zeta + \zeta_y w_\zeta)] \end{aligned}$$

$$\begin{aligned} R_{zmom} = & \zeta_x \partial_\zeta [\mu(\zeta_z u_\zeta + \zeta_x w_\zeta)] \\ & + \zeta_y \partial_\zeta [\mu(\zeta_z v_\zeta + \zeta_y w_\zeta)] \\ & + \zeta_z \partial_\zeta [\mu(\zeta_z w_\zeta + \zeta_z w_\zeta) + \lambda(\zeta_x u_\zeta + \zeta_y v_\zeta + \zeta_z w_\zeta)] \end{aligned}$$

$$\begin{aligned} R_{ener} = & \zeta_x \partial_\zeta [\kappa Pr^{-1}(\gamma - 1)^{-1} \zeta_x (a^2)_\zeta + u\lambda(\zeta_x u_\zeta + \zeta_y v_\zeta + \zeta_z w_\zeta) \\ & + u\mu(\zeta_x u_\zeta + \zeta_x u_\zeta) + v\mu(\zeta_y u_\zeta + \zeta_x v_\zeta) + w\mu(\zeta_z u_\zeta + \zeta_x w_\zeta)] \\ & + \zeta_y \partial_\zeta [\kappa Pr^{-1}(\gamma - 1)^{-1} \zeta_y (a^2)_\zeta + v\lambda(\zeta_x u_\zeta + \zeta_y v_\zeta + \zeta_z w_\zeta) \\ & + u\mu(\zeta_y u_\zeta + \zeta_x v_\zeta) + v\mu(\zeta_y v_\zeta + \zeta_y v_\zeta) + w\mu(\zeta_z v_\zeta + \zeta_y w_\zeta)] \\ & + \zeta_z \partial_\zeta [\kappa Pr^{-1}(\gamma - 1)^{-1} \zeta_z (a^2)_\zeta + w\lambda(\zeta_x u_\zeta + \zeta_y v_\zeta + \zeta_z w_\zeta) \\ & + u\mu(\zeta_z u_\zeta + \zeta_x w_\zeta) + v\mu(\zeta_z v_\zeta + \zeta_y w_\zeta) + w\mu(\zeta_z w_\zeta + \zeta_z w_\zeta)] \end{aligned}$$

where $\lambda = -(2/3)\mu$. Simply replace ζ with η if η is to be used as the direction away from the body surface.

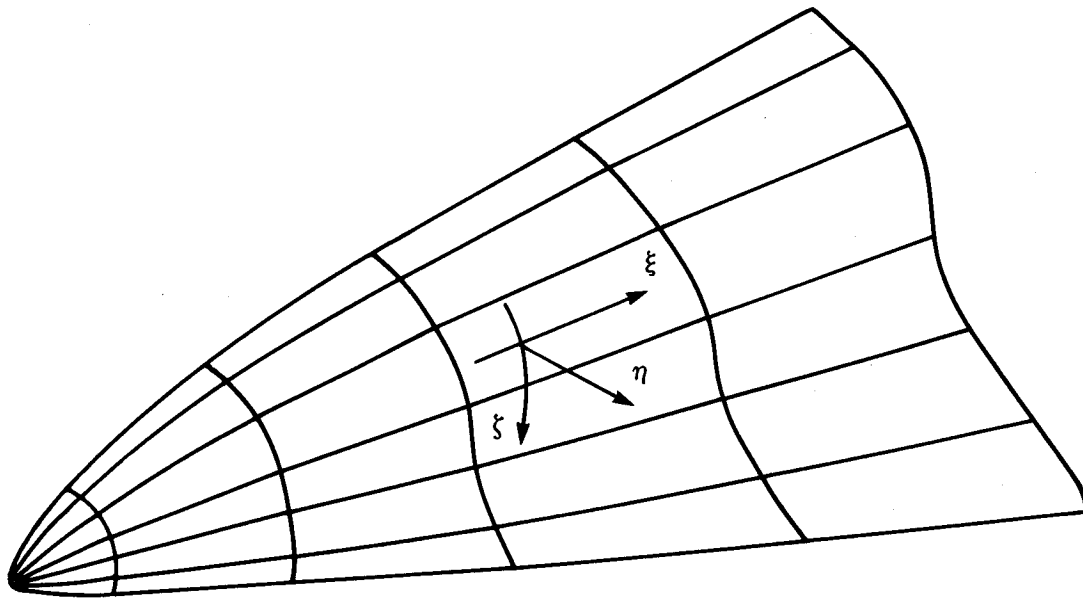


Figure 1.- Sketch showing curvilinear coordinates on $\eta = \text{constant}$ surface.

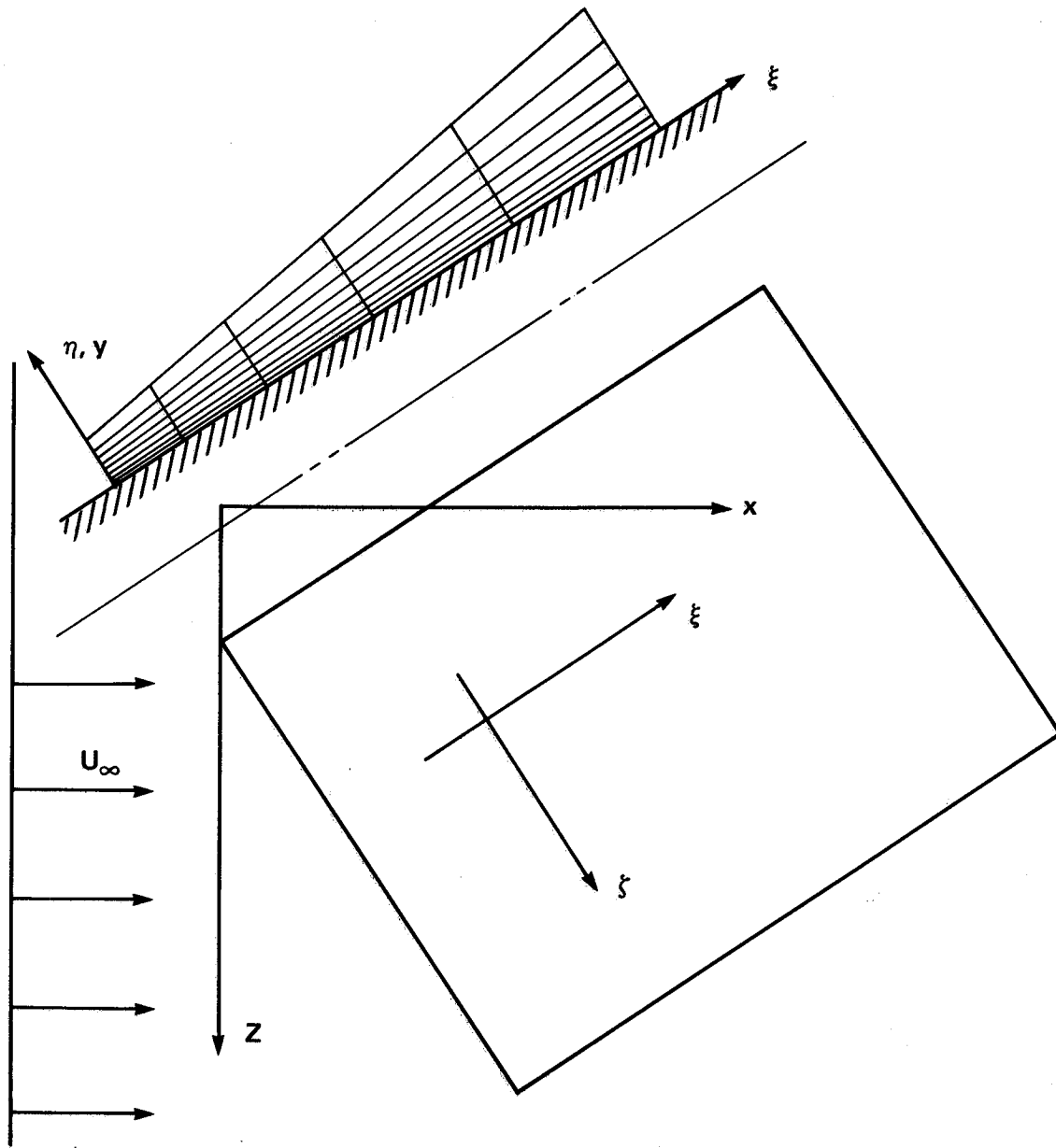


Figure 2.- Orientation of a simple rotated stretched grid used to verify the generalized boundary-layer code on a flat-plate Blasius flow.

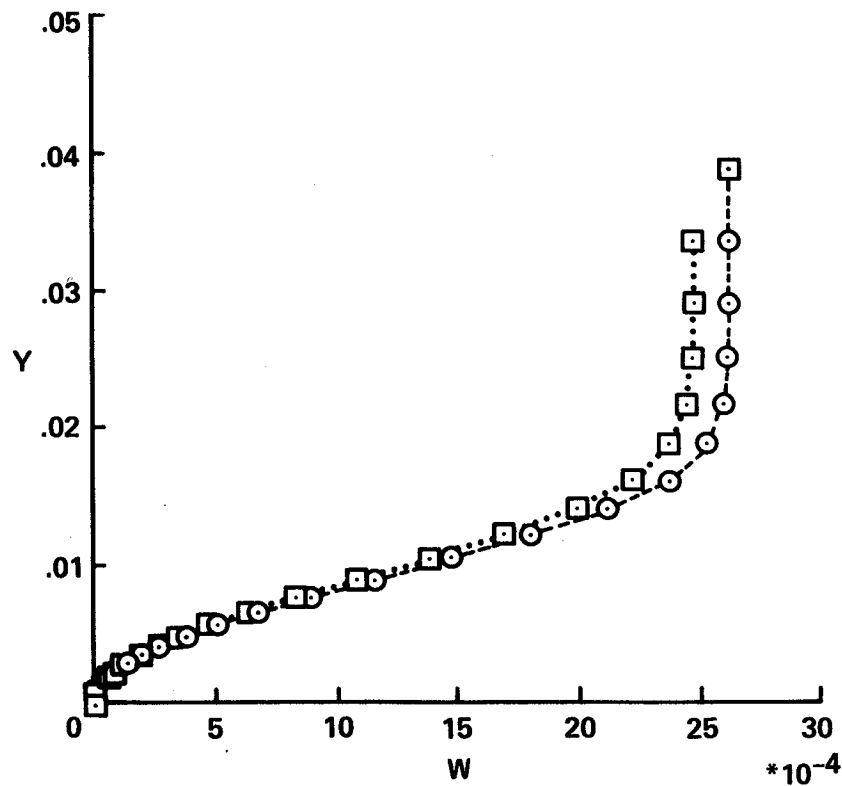
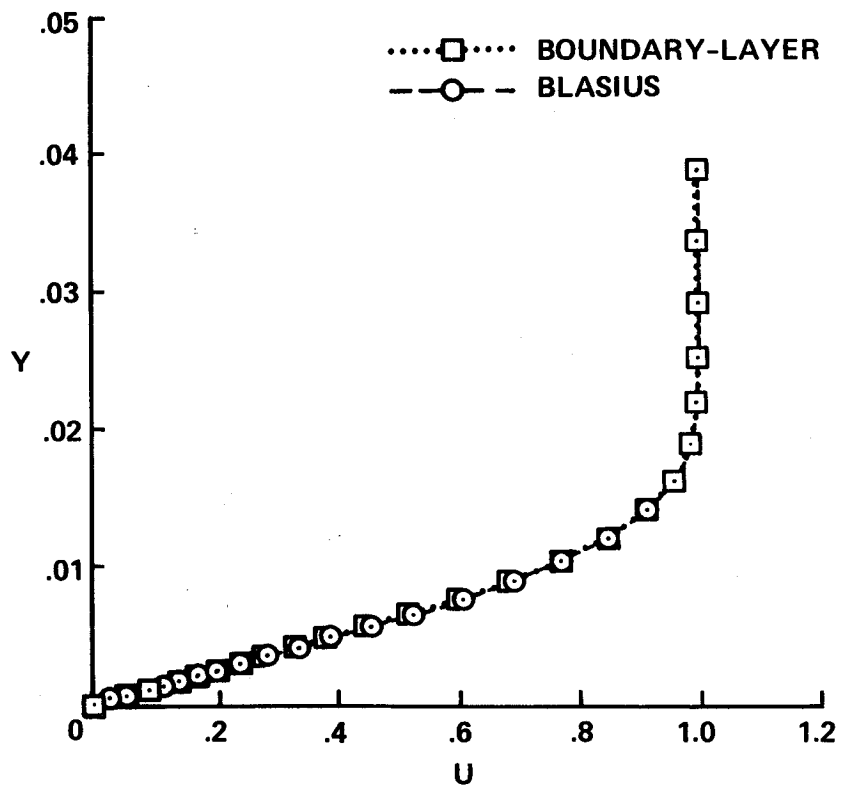


Figure 3.- Computed velocity profiles for u and v taken at the center of a small grid compared with the Blasius solution; no rotation and $Pr = 1$.

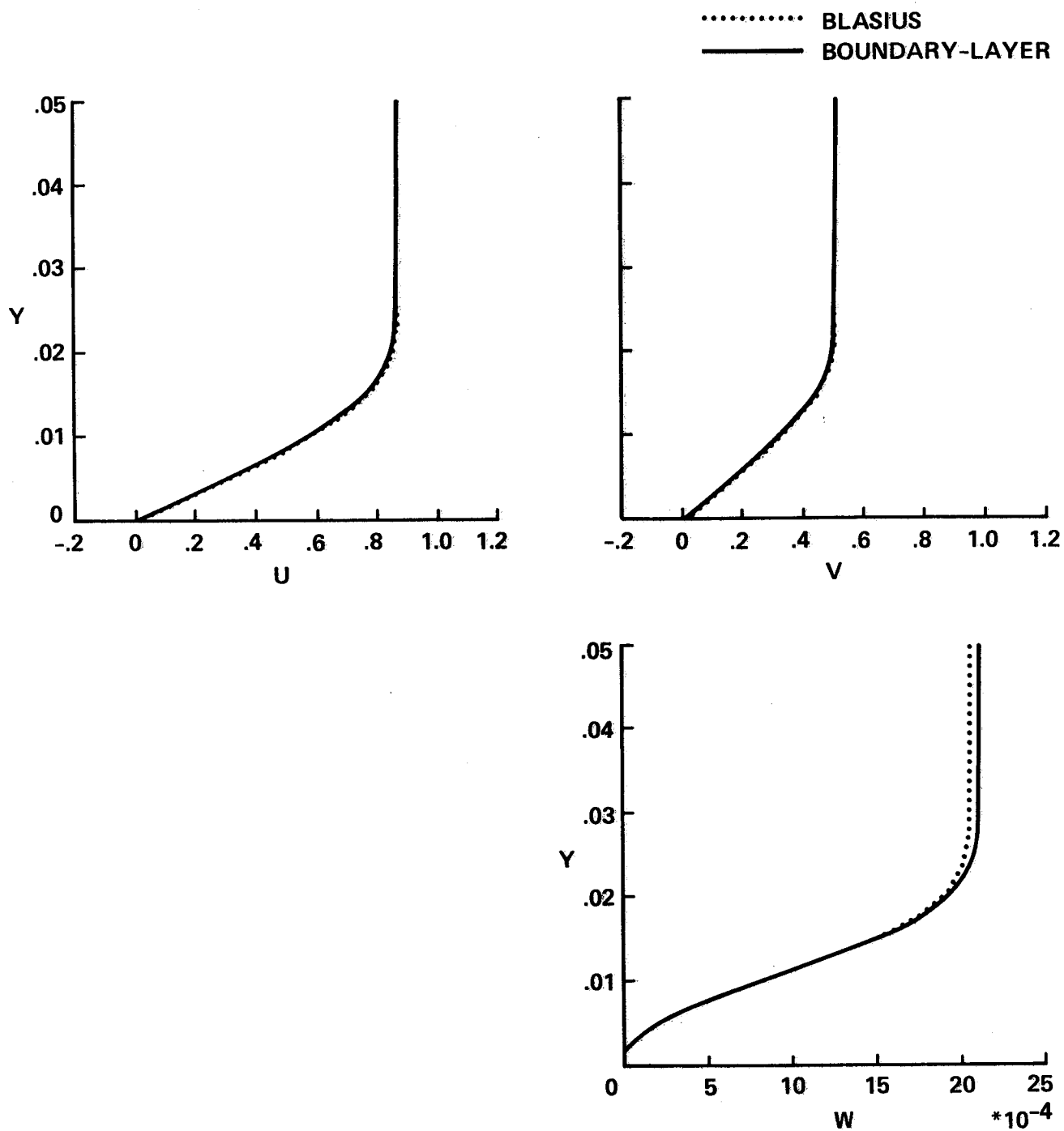


Figure 4.- Computed velocity profiles for u , v , and w taken at the center of a small grid compared with the Blasius solution; grid rotated by 30° and $Pr = 1$.

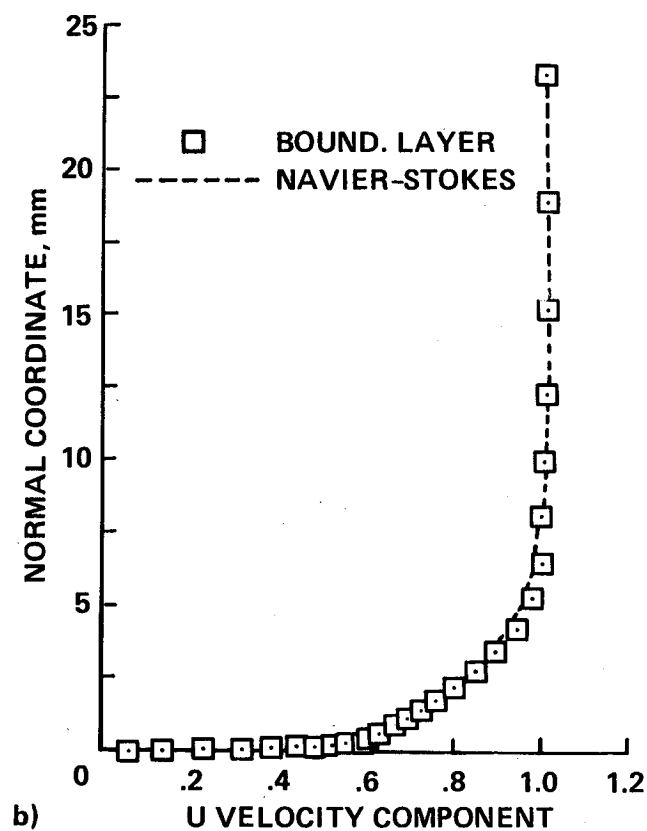
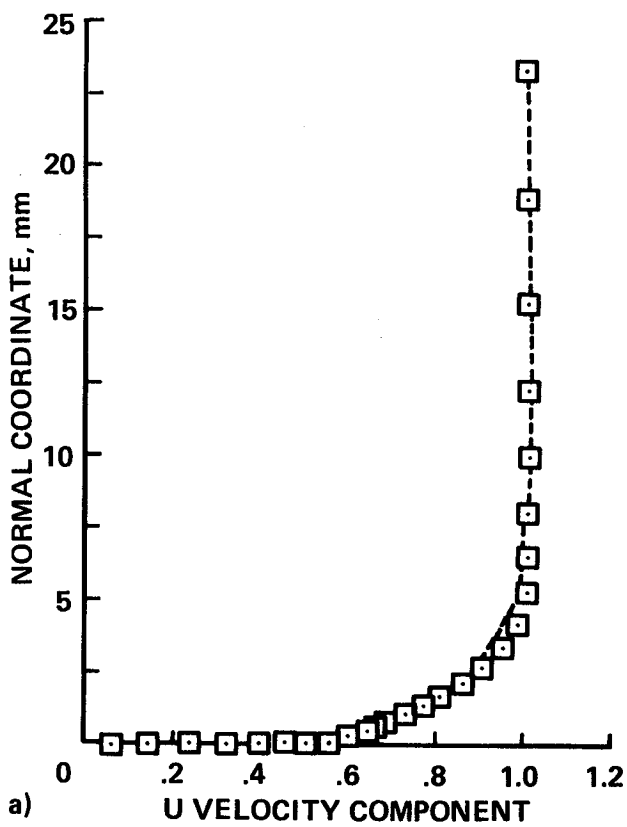
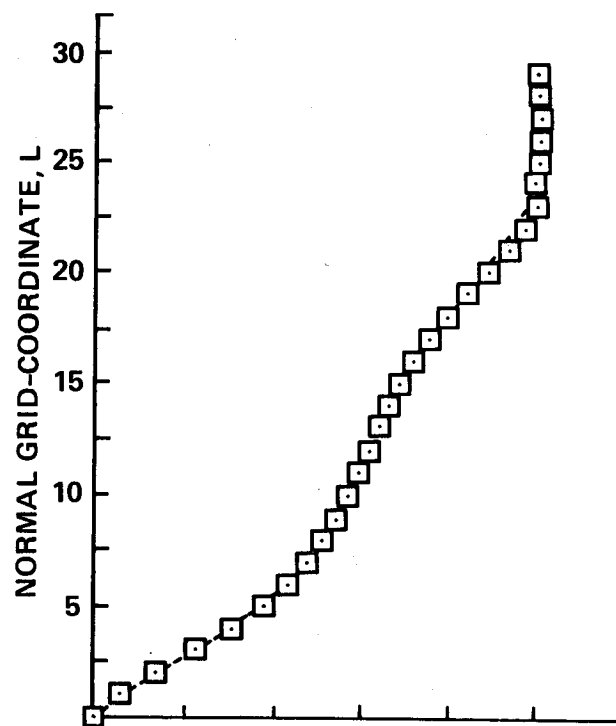
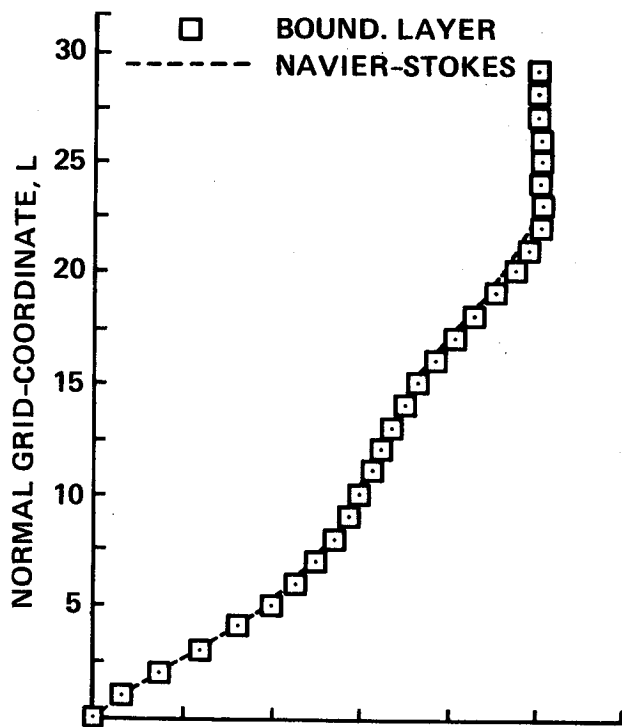


Figure 5.- Transformed and physical plane computed boundary-layer velocity profiles compared to computed thin-layer Navier-Stokes profiles for a 10° -angle-of-attack prolate spheroid at station $x/L = 0.48$ and circumferential locations of a) $\phi = 0^\circ$, b) $\phi = 60^\circ$.

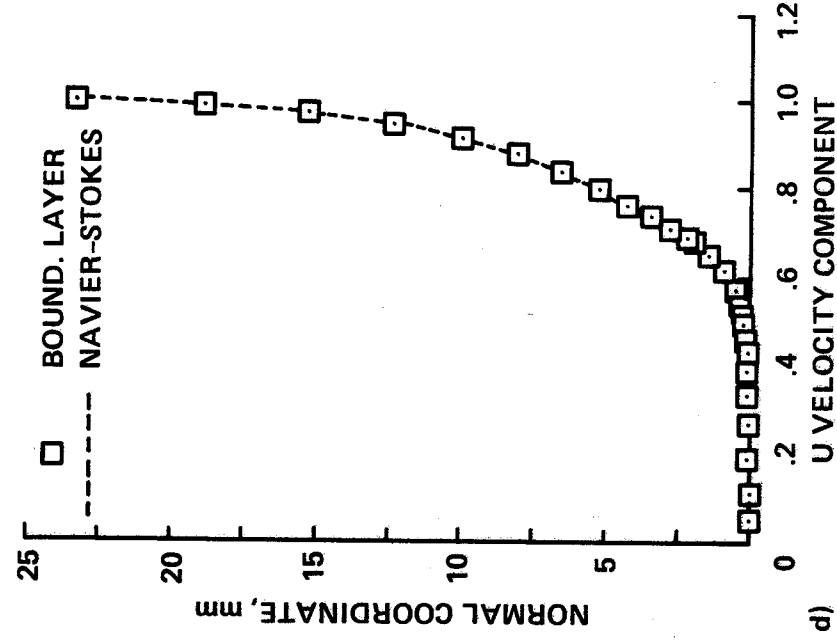
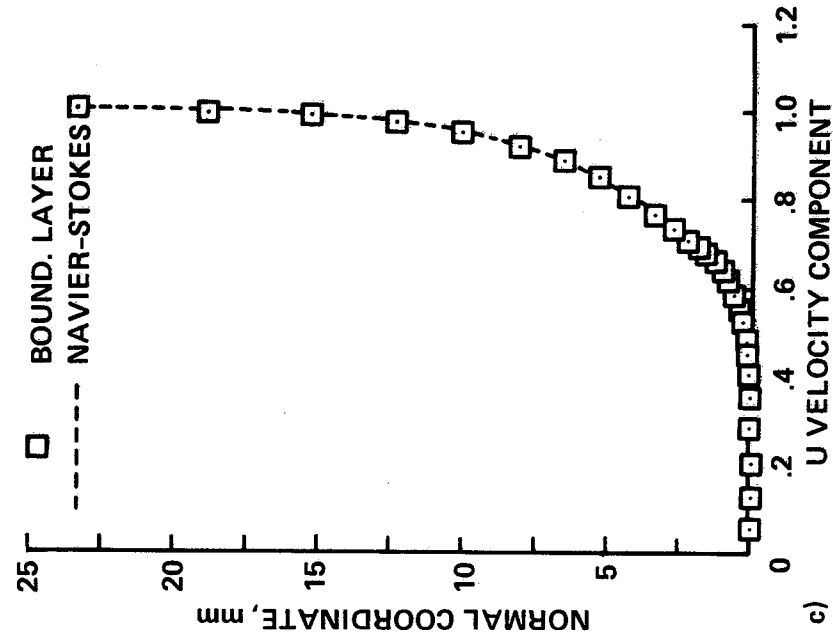
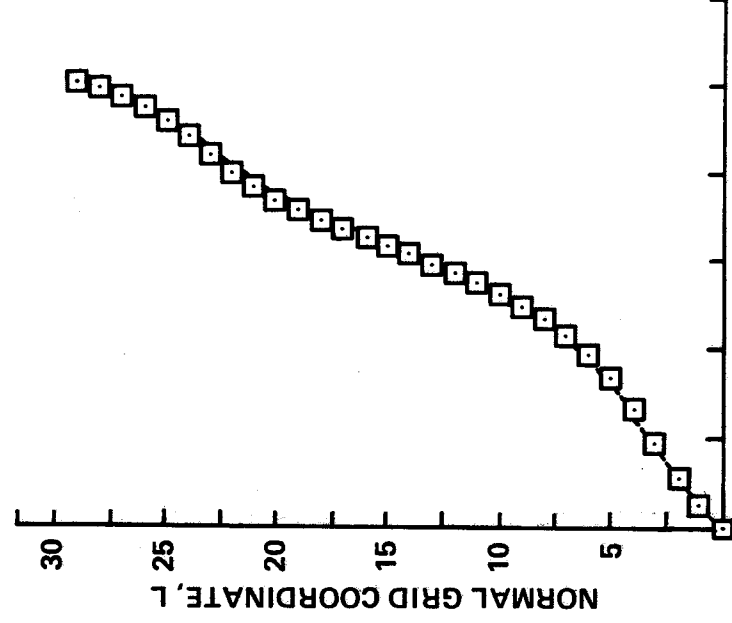
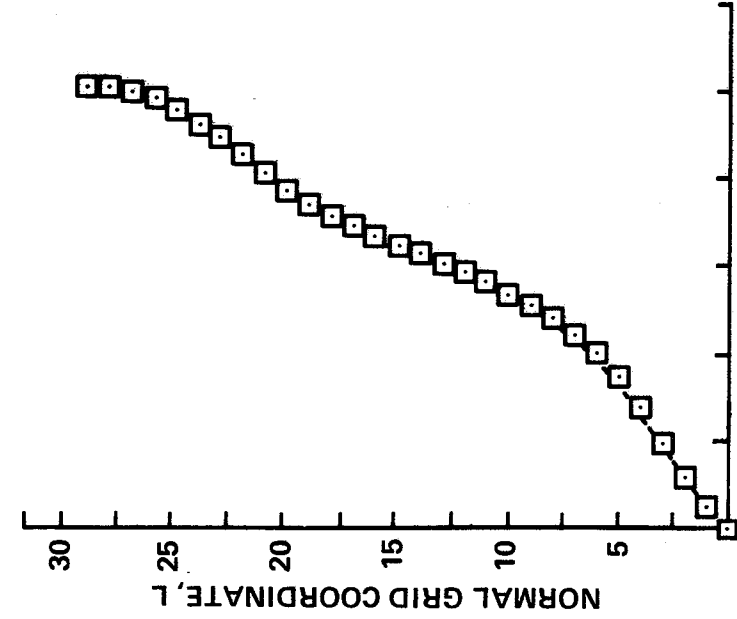


Figure 5.- Concluded. c) $\phi = 120^\circ$, and d) $\phi = 180^\circ$ (leeward).

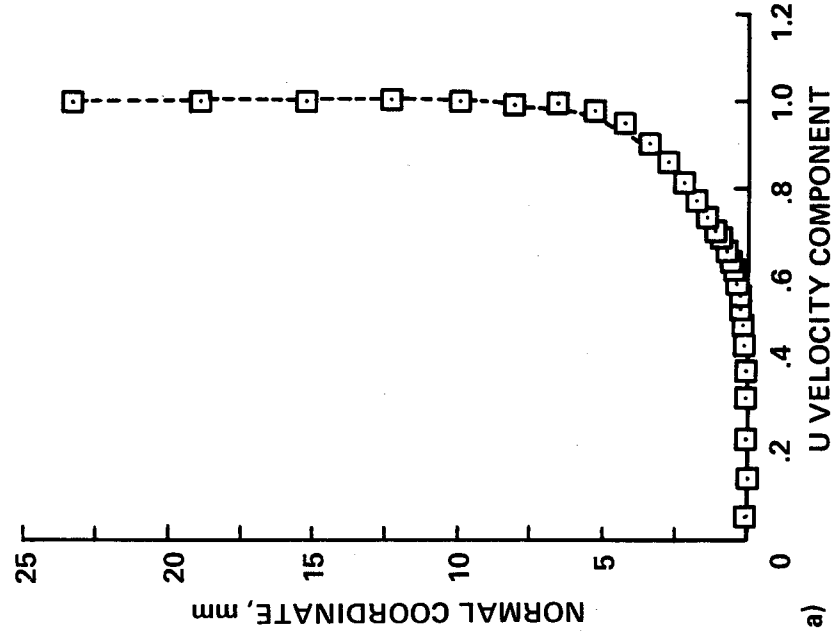
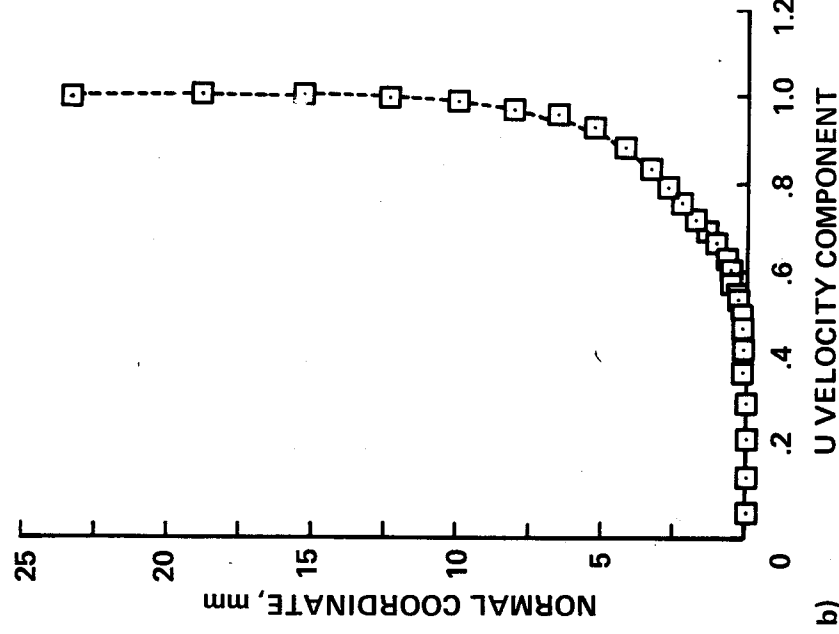
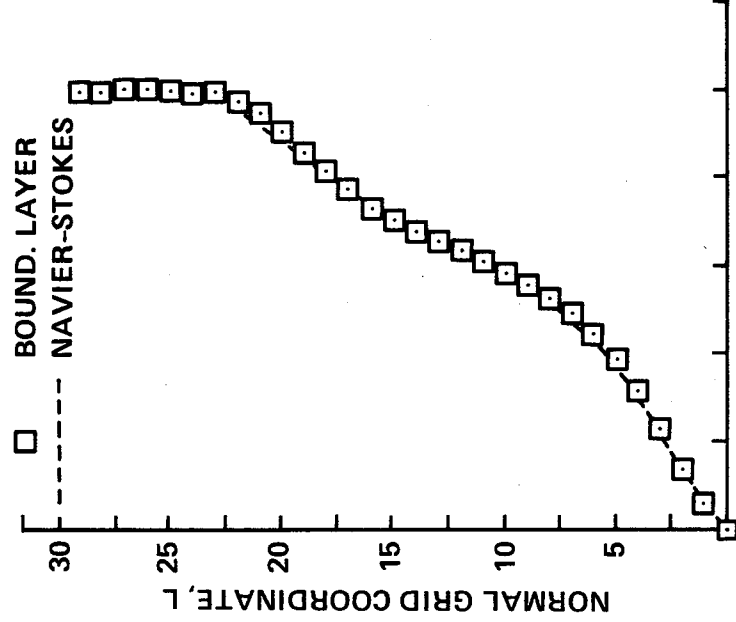
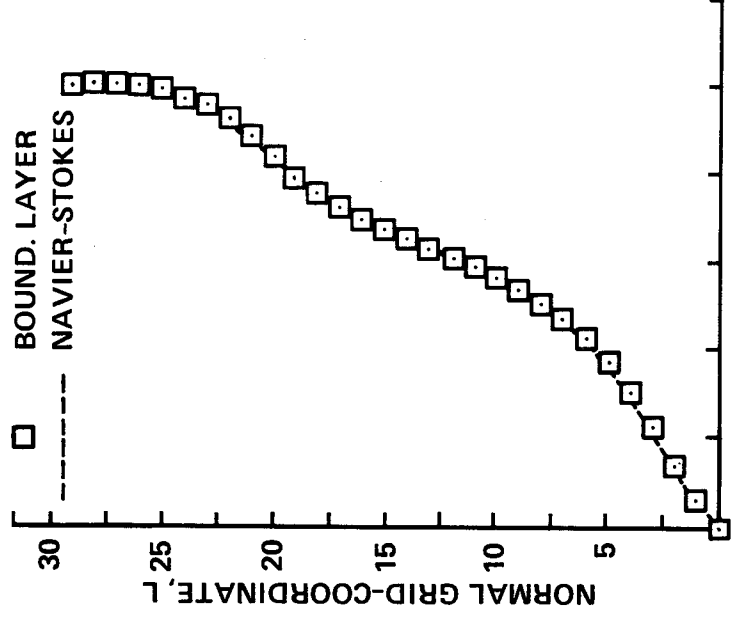


Figure 6.- Computed boundary-layer velocity profiles for transformed and physical plane compared to computed thin-layer Navier-Stokes profiles for a 10° -angle-of-attack prolate spheroid at station $x/L = 0.64$ and circumferential locations of a) $\phi = 0^\circ$, b) $\phi = 60^\circ$.

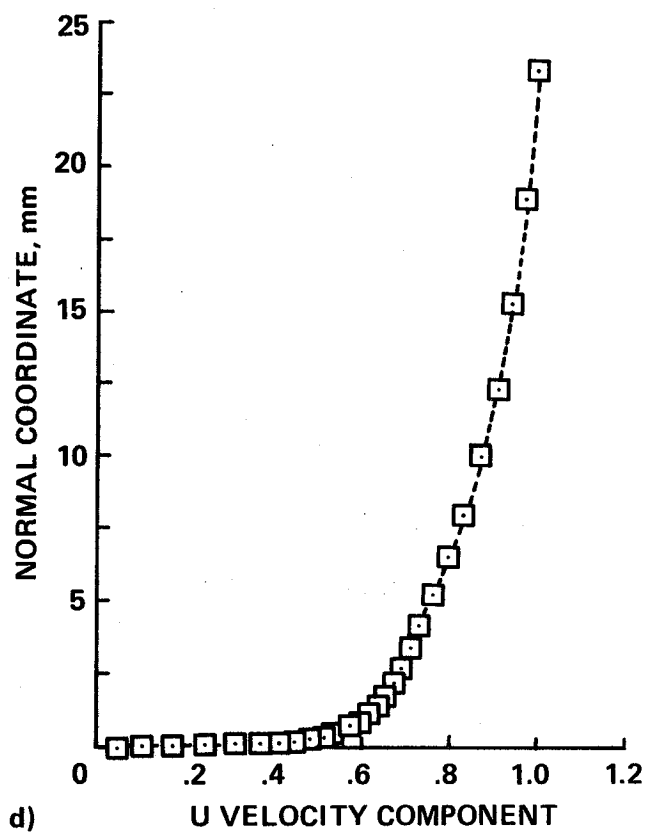
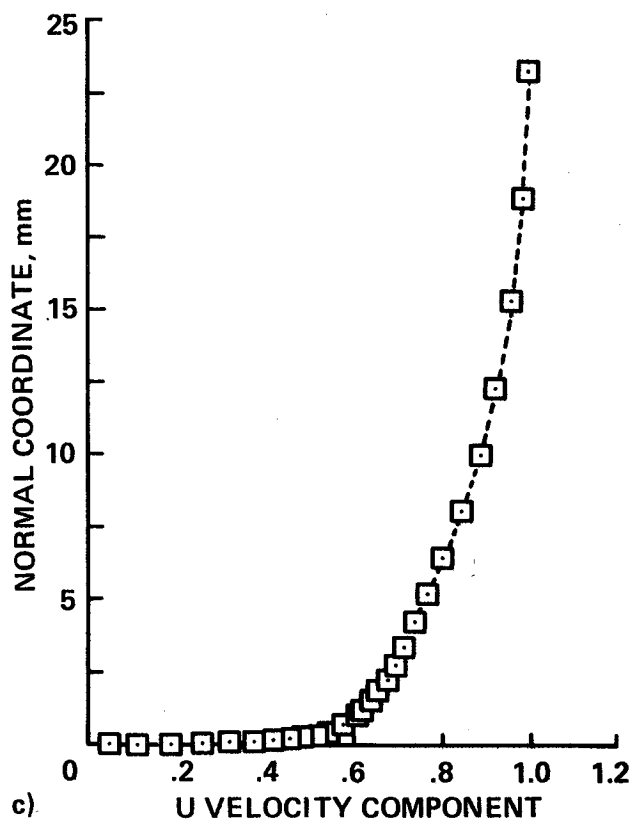
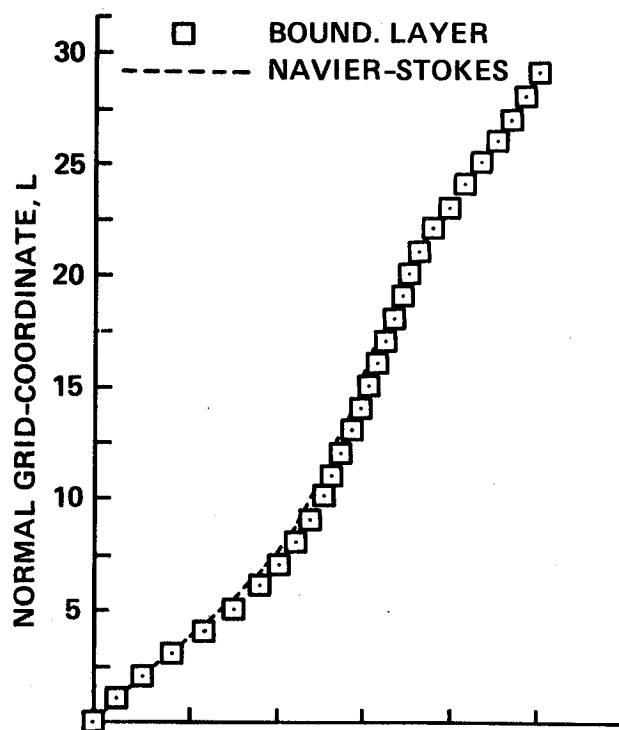
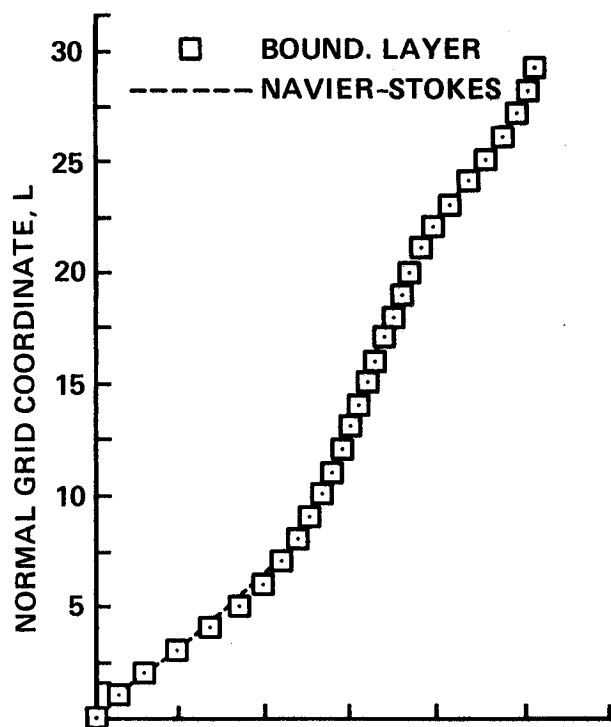


Figure 6.- Concluded. c) $\phi = 120^\circ$, and d) $\phi = 180^\circ$ (leeward).

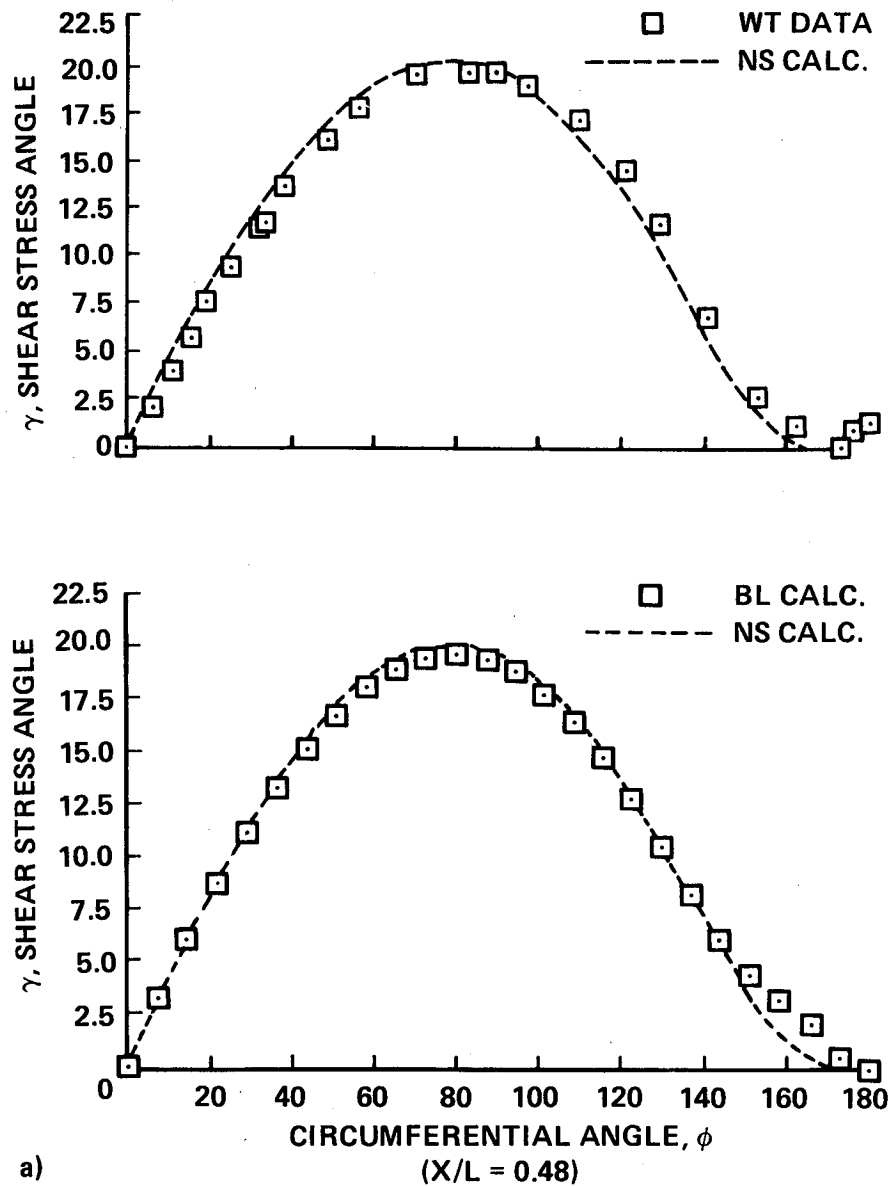


Figure 7.- Wall-turning angles versus circumferential angle ϕ for the boundary-layer and thin-layer Navier-Stokes computations compared with experiment for a 10° -angle-of-attack prolate spheroid at axial stations
a) $x/L = 0.48$.

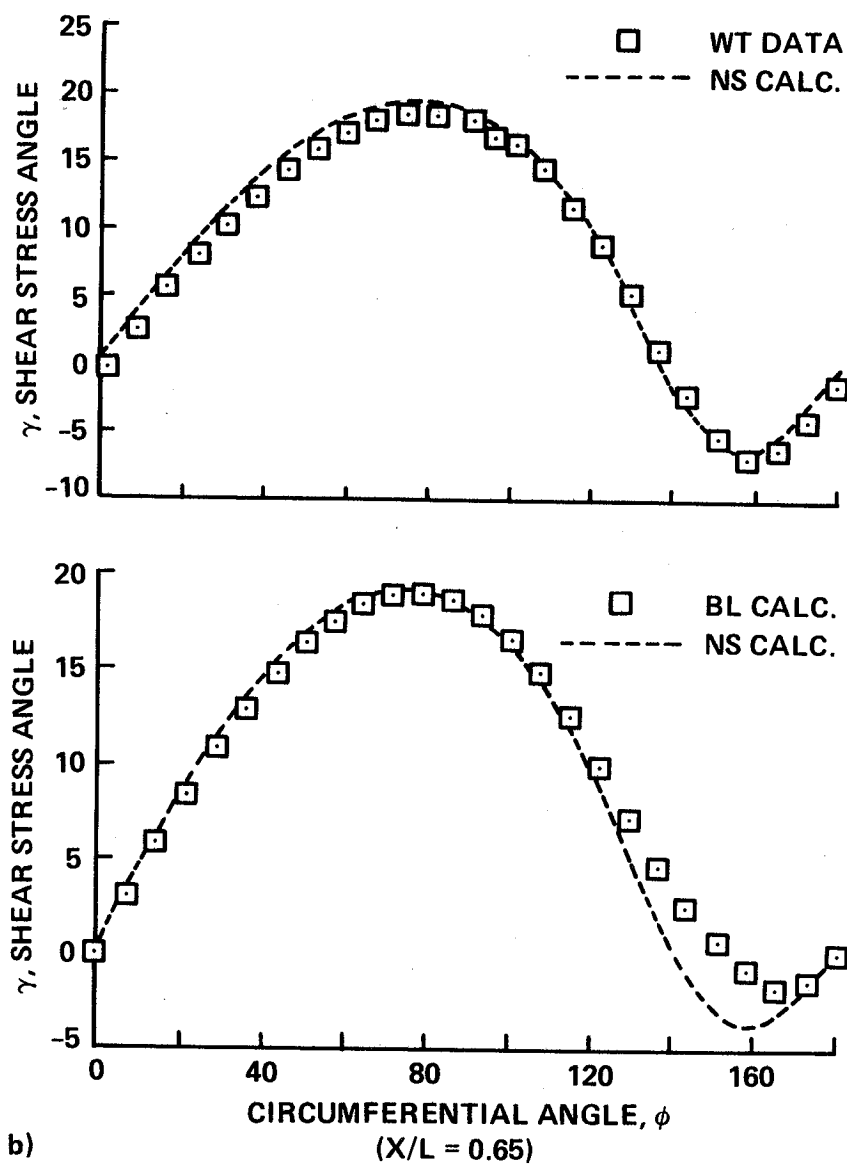


Figure 7.- Concluded. b) $x/L = 0.65$.

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16. Abstract This is a working paper in which a formulation is given for solving the boundary-layer equations in general body-fitted curvilinear coordinates while retaining the original Cartesian dependent variables. The solution procedure does not require that any of the coordinates be orthogonal, and much of the software developed for many Navier-Stokes schemes can be readily used. A limited number of calculations have been undertaken to validate the approach.					
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